Semester - III

Differential and Vector Calculus

Unit-I

1. If $u = x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right) - y^2 \cdot \tan^{-1}\left(\frac{x}{y}\right)$ and $xy \neq 0$ prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

2. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and $x^2+y^2+z^2 \neq 0$ show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

3. If $u = e^{xyz}$ S.T $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}$

4. If $u = \log(x^3+y^3+z^3-3xyz)$ S.T

(i) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{-3}{(x+y+z)^2}$.

5. If $u = \log(\tan x + \tan y + \tan z)$. P.T

$$(\sin 2x)\frac{\partial u}{\partial x} + (\sin 2y)\frac{\partial u}{\partial y} + (\sin 2z)\frac{\partial u}{\partial z} = 2.$$

6. If $u = 3(lx+my+nz)^2 - (x^2+y^2+z^2)$ and $l^2+m^2+n^2 = 1$

S.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

7. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ If $u = \sin^{-1}\left(\frac{x}{y}\right)$

8. If $z = \tan^{-1}\left(\frac{y}{x}\right)$ verify that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

9. If $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$ verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

10. If $u = \log(x^2 + y^2 + z^2)$ P.T $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\cdot\frac{\partial^2 u}{\partial x \partial y}$

11. If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$ S.T $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1+x^2+y^2)^{3/2}}$

12. If $u = x^2(y-z) + y^2(z-x) + z^2(x-y)$ P.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

13. If $u = \log_e \sqrt{x^2+y^2+z^2}$ P.T $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$.

14. If $u = e^x(x\cos y - y\sin y)$ S.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

15. If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

16. State and Prove Euler's theorem on Homogeneous functions.

17. If $z = f(x,y)$ is a Homogeneous function of x, y of degree n, S.T

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

18. If $u = \cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ s.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{4}\sin 2u = 0$.

19. If $u = \text{Tan}^{-1}\frac{x^3+y^3}{x-y}$, $x \neq y$ s.T

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \cdot \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$

20. If $u = \log\left[\frac{x^4+y^4}{x+y}\right]$ show by Euler's theorem that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$.

21. Verify Euler's theorem for

(i) $z = ax^2 + 2hxy + by^2$ (ii) $z = \sin^{-1}\frac{x}{y} + \text{Tan}^{-1}\frac{y}{x}$

22. If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$ s.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \text{Tan}\, u$.

23. If $u = \sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ s.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

24. If $z = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ s.T $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\cot z$.

25. If $u = \text{Tan}^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ s.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.

26. If $V = r^m$ where $r^2 = x^2+y^2+z^2$ s.T

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$$

Unit-II

1. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$

2. If $z = f(x,y)$ P.T If $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$ then
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

3. If $H = f(y-z, z-x, x-y)$ P.T $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0$.

4. P.T if $y^3 - 3ax^2 + x^3 = 0$ then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

5. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$

6. If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^{v}$ find $\frac{\partial z}{\partial v}$.

7. Find $\frac{dy}{dx}$ If (i) $x\sin(x-y) - (x+y) = 0$
(ii) $(\cos x)^y - (\sin y)^x = 0$
(iii) $x^y = y^x$

8. Find $\frac{d^2y}{dx^2}$ If (i) $x^3 + y^3 = 3axy$ (ii) $x^5 + y^5 = 5a^3x^2$

9. Expand $f(x,y) = x^2 + xy - y^2$ by Taylor's theorem in powers of $(x-1)$ and $(y+2)$

10. Obtain Taylor's formula for $f(x,y) = \cos(x+y)$ where $n=3$ at $(0,0)$

(11) Discuss the Maximum & Minimum values of $u = x^3y^2(1-x-y)$

(12) S.T Minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$.

(13) Find a point within a triangle such that sum of squares of the distances from three vertices is Minimum.

(14) Discuss the Maximum & Minimum values of
(i) $u = x^3+y^3-3axy$ (ii) $u = ax^2y^2 - x^4y^2 - x^3y^3$.

(15) Find three positive numbers whose sum is 30 and whose Product is Maximum.

(16) Find Maxima and Minima of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

17. Determine the Maxima and Minima of $x^2+y^2+z^2$ when $ax^2+by^2+cz^2=1$

18. Find the minimum value of $x^2+y^2+z^2$ given that $ax+by+cz=p$.

19. Determine the minimum value of $x^2+y^2+z^2$ subject to the condition $x+2y-4z=5$.

20. P.T $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the function

$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ If $(x,y) \neq (0,0)$

$f(0,0) = 0$.

Vector Calculus

Unit - III

1. Evaluate line integral $\int_C F.dr$ where $F = (5z^2, 2x, x+2y)$ and the curve 'c' is given by $x=t$, $y=t^2$, $z=t^2$, $0 \le t \le 1$

2. Evaluate $\int_C F.dr$ where $F = (5z^2, 2x, x+2y)$ along the straight line joining the points $(0,0,0)$ and $(1,1,1)$.

3. Find the line integral of the vector field $u = (y^2, x, z)$ along the curve given by $z = y = e^x$ from $x=0$ to $x=1$.

4. Find the line integral $\oint_C r \times dr$ where the curve 'c' is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ taken in anticlock wise direction.

5. Evaluate $\int_C (x+y^2)\,dr$ where 'c' is the Parabola $y=x^2$ in the plane $z=0$ connecting the points $(0,0,0)$ and $(1,1,0)$.

6. Evaluate the surface integral of $u = (xy, x, x+y)$ over the surface 's' defined by $z=0$ with $0 \le x \le 1$, $0 \le y \le 2$ with the normal 'n' directed in the positive z direction.

(7) Find the surface integral of $u \cdot n$ over 'S' where 'S' is the part of the surface $z = x + y^2$ with $z < 0$ and $x > -1$ 'u' is the vector field $u = (2y + x, -1, 0)$ and 'n' has a negative 'z' component.

(8) Find the volume of the section of the cylinder $x^2 + y^2 = 1$ that lies between the planes $z = x + 1$ and $z = -x - 1$.

(9) A cube $0 \leq x, y, z \leq 1$ has a variable density given by $\rho = 1 + x + y + z$ what is the total mass of the cube?

(10) Find the volume of the tetrahedron with vertices at $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

(11) Define Line Integral, Surface Integral, Volume Integral.

(12) Evaluate the surface integral of $u = (y, x^2, z^2)$ over the surface 'S', where 'S' is the Triangular surface on $x = 0$ with $y > 0$, $z > 0$, $y + z \leq 1$ with the normal 'n' directed in the positive x direction.

(13) If S is the entire xy Plane, Evaluate the integral $I = \iint_S e^{-x^2 - y^2} \, ds$ by transforming the integral into Polar coordinates.

(14) Find $\iint_S u.n\, ds$, vector field $u = (x, z-y)$ over the curved surface of the cylinder $x^2+y^2=1$ lies between $z=0$ and $z=1$.

(15). Find the volume integral of the scalar field $\phi = x^2+y^2+z^2$ over the region V specified by $0 \le x \le 1$, $1 \le y \le 2$, $0 \le z \le 3$.

(16) Evaluate $\iint_S F.N\, ds$ where $F = zi+xj-3y^2zK$ and 'S' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

(17) Evaluate $\iint_S F.N\, ds$ where $F = 18zi-12j+3yK$ and 'S' is the part of the plane $2x+3y+6z=12$ located in the first octant.

(18) If $F = 2xzi-xj+y^2K$ Evaluate $\iiint_V F.dV$ where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$ $z=x^2$, $z=4$.

Unit - IV

1. Find the unit normal 'n' to the surface $x^2+y^2-z=0$ at the point $(1,1,2)$.

2. Find the directional derivative of the scalar field $f = 2x+y+z^2$ in the direction of the vector $(1,1,1)$ and also evaluate this at the origin.

3. Show that the vector field $F=(2x+y, x, 2z)$ is conservative.

4. Find the gradient of the scalar field $f=xyz$ and evaluate it at the point $(1,2,3)$. Hence find the directional derivative of f at this point in the direction of the vector $(1,1,0)$.

5. Find the unit normal to the surface $y=x+z^3$ at the point $(1,2,1)$.

6. Find the angle between the surfaces of the sphere $x^2+y^2+z^2=2$ and the cylinder $x^2+y^2=1$ at a point where they intersect.

7. Determine whether or not the vector field $F=(\sin y, x, 0)$ is conservative.

8. Find the divergence of the vector field $u=r$.

9. The vector field $u=(xy, z+x, y)$ calculate $\nabla \times u$ and find the points where $\nabla \times u = 0$.

10. Find the gradient $\nabla\phi$ and the Laplacian $\nabla^2\phi$ for the scalar field $\phi = x^2+xy+yz^2$.

11. Find the unit normal to the surface $xy^2+2yz=4$ at the point $(-2,2,3)$.

12. Find both the divergence and curl of the vector fields. (a) $u=(y,z,x)$ (b) $u=(xyz, z^2, x-y)$.

13. For what values if any of the constants a and b is the vector field $u=(y\cos x+axz,\ b\sin x+z,\ x^2+y)$ irrotational?

14. Show that $u=(y^2z,\ -z^2\sin y+2xyz,\ 2z\cos y+y^2x)$ is irrotational.

15. Find divF, curlF where $F=\text{grad}(x^3+y^3+z^3-3xyz)$

16. (i) Show that $3y^4z^2\,i+4x^3z^2\,j-3x^2y^2\,k$ is solenoidal.

(ii) If $f=(x+3y)\,i+(y-2z)\,j+(x+pz)\,k$ is a solenoidal find p.

17. (i) S.T $\text{curl}(\text{grad}\,\phi)=0$ (ii) S.T $\text{div}(\text{curl}\,f)=0$.

LAQ (18) State and Prove Divergence theorem. (Gauss divergence theorem)

LAQ (19) State and Prove Green's theorem

LAQ (20) State and Prove stokes theorem.