Numerical Differentiation and Integration

① ⓐ The forward difference formula of $f(x)$ at $x = x_0$ is

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}, \quad \text{when } h > 0.$$

ⓑ The Backward difference formula of $f(x)$ at $x = x_0$ is

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h}, \quad \text{when } h < 0.$$

ⓒ Error bound $= \frac{M|h|}{2} = \frac{|h \cdot f''(\xi)|}{2}$. where M is a bound on $|f''(x)|$ for x b/w x_0 & x_0+h.

② **Three - Point formulae :-**

ⓐ **Three - Point End Point formula :-**

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

(Here ξ_0 lies between x_0 and x_0+2h), Error bound $= \frac{h^2}{3} f^{(3)}(\xi_0)$

Here $\boxed{h = x_{i+1} - x_i}$, for $i = 0, 1, 2, \ldots$

ⓑ **Three - Point Mid Point formula :-**

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right].$$

(Here ξ_1 lies between $x_0 - h$ and x_0+h), Error bound $= \frac{h^2}{6} f^{(3)}(\xi_1)$

③ **Five - Point formulae :-**

ⓐ **Five - Point End Point formula :-**

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right].$$

(Here ξ lies between x_0 and x_0+4h)

Error bound $= \frac{h^4}{5} f^{(5)}(\xi)$

(b) Five Point Mid Point formula :-

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right].$$

(where ξ lies between x_0-2h and x_0+2h)

Error bound $= \frac{h^4}{30} f^{(5)}(\xi)$.

(4) Second derivative Mid Point formula :

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right].$$

(Here ξ lies between x_0-h and x_0+h).

Error bound $= \frac{h^2}{12} f^{(4)}(\xi)$

Exercise - 4.1

(2) (b) Using forward and backward difference formula to determine each missing entry in the following table:

x	$f(x)$	$f'(x)$
1.0	1.0000	
1.2	1.2625	
1.4	1.6595	

Sol:- forward difference formula $f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} \rightarrow (1)$

Here $h = 1.2 - 1 = 0.2$.

Let $x_0 = 1$ from (1) $f'(1) = \frac{f(1+0.2) - f(1)}{0.2} = \frac{f(1.2) - f(1)}{0.2}$

$= \frac{1.2625 - 1}{0.2}$

$\boxed{f'(1) = 1.3125}$

Let $x_0 = 1.2$ from ① $f'(1.2) = \frac{f(1.2+0.2) - f(1.2)}{0.2} = \frac{f(1.4) - f(1.2)}{0.2}$

$$= \frac{1.6595 - 1.2621}{0.2}$$

$$\boxed{f'(1.2) = 1.987}$$

Back-ward difference formula, $f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$.

Let $x_0 = 1.4$

$$f'(0.4) = \frac{f(1.4) - f(1.4 - 0.2)}{0.2}$$

$$= \frac{f(1.4) - f(1.2)}{0.2} = 1.987$$

$\therefore$

x	$f(x)$	$f'(x)$
1.0	1	1.3125
1.2	1.2625	1.987
1.4	1.6595	1.987.

④ ⓑ From above problem (data) for the function $f(x) = x^2 \ln x + 1$ Calculate Actual Error and Error bounds using Error formulas.

Sol:- $f(x) = x^2 \ln x + 1 \Rightarrow f'(x) = x^2\left(\frac{1}{x}\right) + \ln x\,(2x)$

$$\boxed{f'(x) = x + 2x \ln x}$$

$$f'(x) = x(1 + 2\ln x)$$

$$f''(x) = x\left(\frac{2}{x}\right) + (1 + 2\ln x)$$

$$\boxed{f''(x) = 3 + 2\ln x}$$

At $x = 1 \Rightarrow f'(1) = 1 + 2(1)\ln(1) = 1 + 2\ln(1) = 1$

At $x = 1.2 \Rightarrow f'(1.2) = 1.2 + 2(1.2)\ln(1.2) = 1.2 + 2.4\ln(1.2) = 1.6376.$

At $x = 1.4 \Rightarrow f'(1.4) = 1.4 + 2(1.4)\ln(1.4) = 1.4 + 2.8\ln(1.4) = 2.3421$

Actual Error = |Actual Value − Approximate value|

At $x=1 \Rightarrow$ Actual Error $= |1 - 1.3125| = 0.3125$

At $x=1.2 \Rightarrow$ Actual Error $= |1.6376 - 1.987| = 0.3494$

At $x=1.4 \Rightarrow$ Actual Error $= |2.3421 - 1.987| = 0.3551$

Error bound :-

$$\text{Error bound} = \frac{M|h|}{2} = \frac{|h \cdot f''(\xi)|}{2}$$ where $\xi \in (x_0, x_0+h)$

$$f''(x) = 3 + 2\ln x$$

At $x=1=x_0$, $h=0.2$, $x_0+h = 1+0.2 = 1.2$, $\therefore (x_0, x_0+h) = (1, 1.2)$

$$\text{Error bound} = \frac{|(0.2)(3+2\ln(1.2))|}{2}$$

(∵ $f''(x)$ is increasing in $x_0+h = 1.2$. So Take $\ln x = \ln x_0+h$)

Error bound $= 0.3365$

At $x=1.2=x_0$, $h=0.2$, $x_0+h = 1.2+0.2 = 1.4$, $\therefore (x_0, x_0+h) = (1.2, 1.4)$

$$\text{Error bound} = \frac{|(0.2)(3+2\ln(1.4))|}{2}$$

Error bound $= 0.3673$

(∵ $f''(x)$ is increase in 1.4)

At $x=1.4=x_0$, $h=-0.2$, $x_0+h = 1.4-0.2 = 1.2$.

$\therefore (x_0, x_0+h) = (1.4, 1.2)$

$$\text{Error bound} = \frac{|(0.2)(3+2\ln(1.4))|}{2}$$

∵ $f''(x)$ is Increase in 1.4

Error bound $= 0.3673$.

∴

x	Actual Error	Error bound
1	0.3125	0.3365
1.2	0.3494	0.3673
1.4	0.3551	0.3673

5 (a) Use the most accurate three-Point formula to determine each missing entry from the following data:

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

Sol:- let $x_0 = 1.1,\ x_1 = 1.2,\ x_2 = 1.3,\ x_3 = 1.4$

$f(1.1) = 9.025013,\ f(1.2) = 11.02318,\ f(1.3) = 13.46374,\ f(1.4) = 16.44465$

$h = x_{i+1} - x_i$ for $i = 0, 1, 2, \ldots$

$h = x_1 - x_0 = 1.2 - 1.1 = 0.1$

$\boxed{h = 0.1}$

For $x_0 = 1.1$ ⇒ By Three-Point End Point formula

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

Take $x_0 = 1.1,\ h = 0.1$

$$f'(1.1) = \frac{1}{2(0.1)}\left[-3f(1.1) + 4f(1.2) - f(1.3)\right]$$

$$= \frac{1}{0.2}\left[-3(9.025013) + 4(11.02318) - (13.46374)\right]$$

$$\boxed{f'(1.1) = 17.7697}$$

For $x_0 = 1.2$ ⇒ By Three-Point Mid Point formula

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$f'(1.2) = \frac{1}{2(0.1)}\left[f(1.2+0.1) - f(1.2-0.1)\right]$$

$$= \frac{1}{0.2}\left[f(1.3) - f(1.1)\right] = \frac{1}{0.2}\left[13.46374 - 9.025013\right]$$

$$\boxed{f'(1.2) = 22.1936}$$

For $x_0 = 1.3$, By Three-Point Mid Point formula.

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$f'(1.3) = \frac{1}{2(0.1)}\left[f(1.3+0.1) - f(1.3-0.1)\right]$$

$$= \frac{1}{0.2}\left[f(1.4) - f(1.2)\right] = \frac{1}{0.2}\left[16.44465 - 11.02318\right]$$

$$\boxed{f'(1.3) = 27.1074}$$

For $x_0 = 1.4 \Rightarrow$ By Three-Point End Point formula.
and $\boxed{h = -0.1}$

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

$$f'(1.4) = \frac{1}{2(-0.1)}\left[-3f(1.4) + 4f(1.4-0.1) - f(1.4-2(0.1))\right]$$

$$= \frac{1}{-0.2}\left[-3f(1.4) + 4f(1.3) - f(1.2)\right]$$

$$= \frac{1}{-0.2}\left[-3(16.44465) + 4(13.46374) - (11.02318)\right]$$

$$\boxed{f'(1.4) = 32.5109}$$

$\therefore$

x	$f(x)$	$f'(x)$
1.1	9.025013	17.7697
1.2	11.02318	22.1936
1.3	13.46374	27.1074
1.4	16.44465	32.5109

7 (a) From the above problem data for the function $f(x) = e^{2x}$ calculate Actual error.

Sol:- Given $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

For $x = 1.1 \Rightarrow f'(1.1) = 2e^{2(1.1)} = 2e^{2.2} = 18.05$

Actual Error = |Actual value − Approximate value|

$= |18.05 - 17.769705|$

$= 0.2803$

For $x = 1.2 \Rightarrow f'(1.2) = 2e^{2(1.2)} = 2e^{2.4} = 22.0464$

Actual Error = |Actual Value − Approximate value|

$= |22.0464 - 22.1936|$

$= 0.1472$

For $x = 1.3 \Rightarrow f'(1.3) = 2e^{2(1.3)} = 2e^{2.6} = 26.9275$

Actual error = |Actual value − Approximate value|

$= |26.9275 - 27.1074|$

$= 0.1799$

For $x = 1.4 \Rightarrow f'(1.4) = 2e^{2(1.4)} = 2e^{2.8} = 32.8893.$

Actual error = |Actual Value − Approximate Value|

$= |32.8893 - 32.5109|$

$= 0.3784$

(10)

(a) Use the five-Point formula find the approximations for each missing entry from the following data:

x	$f(x)$	$f'(x)$
1.05	−1.709847	
1.10	−1.373823	
1.15	−1.119214	
1.20	−0.9160143	
1.25	−0.7470223	
1.30	−0.6015966	

Sol:- $h = 0.05$, $f(1.05) = -1.709847$, $f(1.10) = -1.373823$

$f(1.15) = -1.119214$, $f(1.20) = -0.9160143$

$f(1.25) = -0.7470223$, $f(1.30) = -0.6015966$

At $x_0 = 1.05$, By Five-Point End Point formula

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right]$$

$$= \frac{1}{12(0.05)}\left[-25f(1.05) + 48f(1.10) - 36f(1.15) + 16f(1.20) - 3f(1.25)\right]$$

$$= \frac{1}{0.6}\left[4.6792\right]$$

$$\boxed{f'(1.05) = 7.7987}$$

At $x_0 = 1.10$, By Five-Point End Point formula.

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right]$$

$$= \frac{1}{12(0.05)}\left[-25f(1.10) + 48f(1.15) - 36f(1.20) + 16f(1.25) - 3f(1.30)\right]$$

$$= \frac{1}{0.6}\left[3.4523\right]$$

$$\boxed{f'(1.10) = 5.7538}$$

At $x_0 = 1.15$, By Five-Point mid-Point formula.

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right]$$

$$= \frac{1}{12(0.05)}\left[f(1.05) - 8f(1.10) + 8f(1.20) - f(1.25)\right]$$

$$= \frac{1}{0.6}\left[2.6996\right]$$

$$\boxed{f'(1.15) = 4.4994}$$

At $x_0 = 1.20$, By Five-Point mid-Point formula.

$$f'(x_0) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right]$$

$$= \frac{1}{12(0.05)}\left[f(1.10) - 8f(1.15) + 8f(1.25) - f(1.30)\right]$$

$$= \frac{1}{0.6}\left[2.2053\right]$$

$$\boxed{f'(1.20) = f'(x_0) = 3.6755}$$

At $x_0 = 1.25$, By Five-Point End Point formula.

$\boxed{h = -0.05}$

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right]$$

$$= \frac{1}{-12(0.05)}\left[-25f(1.25) + 48f(1.20) - 36f(1.15) + 16f(1.10) - 3f(1.05)\right]$$

$$= \frac{-1}{0.6}\left[-1.8531\right]$$

$$\boxed{f'(1.25) = 3.0884}$$

At $x_0 = 1.30$, By Five-Point End Point formula.

$\boxed{h = -0.05}$

$$f'(x_0) = \frac{1}{12h}\left[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\right]$$

$$= \frac{1}{12(-0.05)}\left[-25f(1.30) + 48f(1.25) - 36f(1.20) + 16f(1.15) - 3f(1.10)\right]$$

$$= \frac{-1}{0.6}\left[-1.6266\right]$$

$$\boxed{f'(1.30) = 2.7110.}$$

18 (a) From the following data use all the appropriate formulae find approximate to $f'(0.4)$ and $f''(0.4)$.

x	0.2	0.4	0.6	0.8	1
$f(x)$	0.9798652	0.9177710	0.808038	0.6386093	0.3843735

Sol:- $h = 0.2$, $x_0 = 0.4$

By forward difference formula, $\boxed{f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h}}$

$$f'(0.4) = \frac{f(0.4+0.2) - f(0.4)}{0.2} = \frac{f(0.6) - f(0.4)}{0.2} = -0.5487$$

By Backward difference formula, $f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h}$

$$f'(0.4) = \frac{f(0.4) - f(0.4-0.2)}{0.2}$$

$$= \frac{f(0.4) - f(0.2)}{0.2}$$

$$\boxed{f'(0.4) = -0.3105}$$

Three point End point formula :-

$$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]$$

$$= \frac{1}{2(0.2)}\left[-3f(0.4) + 4f(0.6) - f(0.8)\right]$$

$$= \frac{1}{0.4}\left[-0.1598\right]$$

$$\boxed{f'(0.4) = -0.3994}$$

By Three Point Mid Point formula.

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$= \frac{1}{2(0.2)}\left[f(0.6) - f(0.2)\right]$$

$$= \frac{1}{0.4}\left[-0.1718\right]$$

$$\boxed{f'(0.4) = -0.4296}$$

we have to evaluate $f''(0.4)$:-

Second derivative mid point formula

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h)\right]$$

$$f''(0.4) = \frac{1}{(0.2)^2}\left[f(0.2) - 2f(0.4) + f(0.6)\right]$$

$$= \frac{1}{0.04}\left[-0.0476\right]$$

$$\boxed{f''(0.4) = -1.1900}$$